UDC 531.533

# ON THE CONSTRUCTION OF MODELS OF CONTINUOUS MEDIA INTERACTING WITH AN ELECTROMAGNETIC FIELD

PMM Vol.43, No.3, 1979, pp. 387-400

L. I. SEDOV and A. G. TSYPKIN
(Moscow)

(Received December 7, 1978)

A general theory of constructing models of continuous media in the presence of interaction between material bodies and electromagnetic field is proposed which takes into account electric currents, polarization, and magnetization, and is based on the use of the fundamental variational equation. A closed system of equations, including the Maxwell equation, the equation of state, which define polarization, magnetization, and internal mechanical stresses, is established for continuous motions under specified external effects. ( As shown in [1, 2], it is possible to obtain from the fundamental variational equation, also, conditions at strong discontinuities). It is shown that for actual phenomena the fundamental variational equation locally reduces to the first and second laws of thermodynamics also in the presence of electromagnetic fields. A number of important aspects (such as the meaning of used partial time derivatives and of tensor component variations; the concept of the electric field energy as a four-dimensional scalar; selection of scalar function for the Lagrangian, fixing of a nonzero functional $\delta W^*$; expressions for the uncompensated heat, for variational and real processes, etc.) that occur in the details of analysis related to the conversion of the first and second laws of thermodynamics to the universal variational equation. Typical specific examples of models of solid and fluid material media reacting with an electromagnetic field are considered.

Recently a considerable number of publications dealt with the construction of models of continuous media, taking into consideration polarization and magnetization phenonomena and the distribution of mobile charges and conduction currents. However one is met, so far, with the absence of rationally substantiated construction of models based on the use of thermodynamic methods with a minimal number of simplest assumptions. It would be advantageous, if the assumptions, that are always necessary, were formulated on the basis of universal physical principles.

It was shown by Sedov [3] as far back as 1965 that for obtaining from the first and second laws of thermodynamics a closed macroscopic system of equations that are satisfied when applied to continuous processes, it is sufficient in the simplest typical cases of reversible processes to specify, in addition to external influences, the internal energies of the field and the material medium in the form of functions of the polarization and magnetization tensor, of the mechanical characteristics of motion and of internal governing thermodynamic parameters. In the considered models this method makes it possible to obtain all equations of state, including those related to polarization and magnetization laws.

A number of publications deals with the development of this theory using the basic variational equation, and with its extension to processes with weak and strong discontinuities, and with the presence of higher derivatives in arguments of the Lagrange function [4–6]. Note that in the variational equations applied to cases with electromagnetic field, the Lagrange function density was not amenable to thermodynamic interpretation.

In the case of actual processes the basic variational equation for a small volume element of medium and electromagnetic field must, according to the basic idea, reduce locally to the complete equation of balances for increments of all forms of energy that are generated in the investigated processes by the interaction between fields and material media. This aspect may be taken as an essential guiding physical indication for the establishment of the form of functionals that appear in the basic variational equation which may, however, contain also additional terms that vanish for actual processes. Such terms can be represented by the elementary influx of energy of gyroscopic nature, or in the case of variational processes, etc., by terms of special form related to irreversibility.

The present paper is primarily devoted to the following topics.

1. Clarification of the problem of local reduction of the basic variational equation to the equation of energy for the system "electromagnetic field-material medium" taken as a unit. The possibility of such reduction was until now doubted. The following discussion of this problem shows that in the case of reversible processes in electromagnetic fields the derivation of the basic variational equation is complicated by the necessity to take into account the interaction between a small volume of field and medium and the adjacent elementary volumes. The analysis of irreversible processes is further complicated by the appearance of additional terms in the expression for the variational increment of uncompensated heat.

2. Derivation of a closed system of equations including the equations of state for the system electromagnetic field-material medium, using the special theory of relativity.

**1. Basic notation and the coordinate system.** Let $x^1$, $x^2$, $x^3$, $x^4 = ct$ be the coordinates in some selected inertial Cartesian reference system of an observer of a four-dimensional pseudo-Euclidean space, $ds^2 = (cdt)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$, $c$ be the speeed of light in vacuum, $t$ be the time, and $\xi^1$, $\xi^2$, $\xi^3$, $\xi^4 = c\tau$ be the Lagrangian coordinates of the medium in a moving accompanying coordinate system frozen in the medium. We assume that by definition $d\tau$ along the world line $\xi^\alpha = \text{const}$ is equal to the increment of the proper time. We denote the covariant components of the metric tensor of the observer's reference system and of the accompanying system by $g_{ij}, g_{ij}\hat{}$ $(g_{44}\hat{} = 1)$ and $ds^2 = g_{ij}\hat{}\, d\xi^i d\xi^j$, respectively.

In what follows the lower case Latin indices run through $1$–$4$, while the lower case Greek letters run through numbers $1$–$3$. Summation is carried out with respect to coinciding upper and lower indices. The superscript $\hat{}$ indicates that the respective components are defined in the accompanying coordinate system.

For $ds \neq 0$ we use the notation

$$u^i = dx^i / d\xi^4 = dx^i / ds$$

for the contravariant four-dimensional dimensionless unit vector of the medium flow velocity, and for the mass density of the medium

$$\rho = \rho_0 (\xi^\mu) \, [\det \| g_{ij}\hat{} - u_i\hat{} u_j\hat{} \| \, ]^{-1/2} \tag{1.1}$$

Density $\rho_e$ of the free electric charges is defined by a similar formula. It will be readily seen that the definition (1.1) implies that the medium mass density $\rho$ and the charge density $\rho_e$ satisfy the four-dimensional equations of continuity

$$\nabla_i (\rho u^i) = 0, \quad \nabla_i (\rho_e u^i) = 0 \tag{1.2}$$

where $\nabla_i$ is the four-dimensional operator of covariant differentiation in any coordinate system.

Below, in addition to the observer's reference system whose basis vectors we denote by $\mathbf{э}_i$ and of the accompanying system with basis vectors $\mathbf{э}_i\hat{}$ in the four-dimensional space, we also introduce locally (*) at every point $M$ of the medium its proper system of coordinates $x^i$ with the basis tetrad $\mathbf{э}_i{}^*$ such that at that point the Christoffel three-index symbol is zero, i.e. $\Gamma_{sj}{}^{*k} = 0$, and the three-dimensional velocity of a point of the medium is zero relative to that point proper coordinate system. The four-dimensional velocities $\mathbf{u}^*$ of points in the proper system are exactly equal the four-dimensional velocity $\mathbf{u}$ of point $M$ of the medium. At points of the medium adjacent to point $M$ on the world line and in space, generally speaking, $\mathbf{u}^* \neq \mathbf{u}$.

Owing to the selection of the accompanying system of coordinates $(g_{44}\hat{} = 1)$ the following equalities are valid at point $M$:

$$\mathbf{u}^* = \mathbf{u} = \mathbf{э}_4{}^* = \mathbf{э}_4\hat{} \tag{1.3}$$

and, consequently, the axis of time $t$ for the proper coordinate system is tangent to the world line of the medium at point $M$. Hence

$$dt^* = d\tau^* = d\tau$$

Since it is not generally possible to reduce $g_{4\alpha}\hat{} (\xi^\alpha, \tau)$ to zero at all points of the medium, it will be readily appreciated that it is not generally possible to satisfy the equality $\mathbf{э}_\alpha{}^* = \mathbf{э}_\alpha\hat{}$ at all points of the medium. That equality can, however, be satisfied at any one point by an appropriate selection of Lagrangian coordinates.

The totality of proper reference systems with reference points $\mathbf{э}_i{}^*$ for all possible points $M$ constitutes a nonholonomic set, in other words, it is not possible to indicate an over-all reference system with the introduced inertial coordinate reference points $\mathbf{э}_i{}^*$.

---

*) In a Riemannian space the localization property is essential, while in the Minkowski space the proper coordinate system can be introduced as the over-all coordinate system at every point $M$ of that space.

Along the world lines of points of the medium we have

$$\frac{\partial \mathbf{u}^*}{d\tau} = 0, \quad \frac{d\mathfrak{s}_i^*}{d\tau} = 0, \quad \frac{d\mathfrak{s}_i^*}{dx^k} = 0 \tag{1.4}$$

$$\frac{d\mathbf{u}}{d\tau} = \mathbf{a}, \quad \frac{d\mathfrak{s}_i^{\wedge}}{d\tau} = \Gamma_{i4}^{\wedge k} \mathfrak{s}_k^{\wedge} = \frac{d\mathbf{u}}{d\xi^i} = \nabla_i^{\wedge} u^{\wedge k} \mathfrak{s}_k^{\wedge} \neq 0$$

where $\mathbf{a}$ is the generally nonzero absolutely definite four-dimensional vector of acceleration of points of the medium.

If the world line is isotropic, then $g_{44}^{\wedge} = 0$ and $d\tau = 0$ and it is possible to obtain equalities analogous to (1.4) in which increments $d\lambda$ of the related parameter $\lambda$ determined along the iostropic world line are to be substituted for $d\tau$.

To define the effects of interaction between the electromagnetic field and the polarizable and magnetizable medium we introduce antisymmetric tensors of the electromagnetic field with components $F_{ij}$ and $H^{ij}$

$$F_{ij} = \begin{Vmatrix} 0 & B^3 & -B^2 & E_1 \\ -B^3 & 0 & B^1 & E_2 \\ B^2 & -B^1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{Vmatrix}, \quad H^{ij} = \begin{Vmatrix} 0 & H_3 & -H_2 & -D^1 \\ -H_3 & 0 & H_1 & -D^2 \\ H_2 & -H_1 & 0 & -D^3 \\ D^1 & D^2 & D^3 & 0 \end{Vmatrix} \tag{1.5}$$

where the notation conforms to that in [1,3,4]. Note that owing to the method of selecting the proper coordinate system for $\mathfrak{s}_i^{\wedge} = \mathfrak{s}_i^*$ the components of tensors $F_{ij}$ and $H^{ij}$ at every one given point are the same in the proper and in the accompanying coordinate systems.

**2. Equations of energy for the electromagnetic field and medium.** The equation of energy for the electromagnetic field in any inertial and, in particular, in the proper coordinate system can be written as

$$\partial W / \partial \tau = -\operatorname{div} \mathrm{S} - F \tag{2.1}$$

where the proper coordinate system $W = (\mathbf{BH} + \mathbf{DE})/8\pi$ is by definition the electromagnetic field energy ($W$ is a three-dimensional scalar), $\mathrm{S} = (c / 4\pi)\ \mathbf{E} \times \mathbf{H}$ is a three-dimensional Poynting vector, and $F$ is the flow of energy form the field to the medium determined by the process of Joule heat emission and of polarization and magnetization of the continuous medium.

It is not difficult to ascertain that, similarly to the kinetic energy in Newtonian mechanics, the quantity $(1/8\pi)\ (\mathbf{BH} + \mathbf{DE})$ depends on the selection of the inertial reference system. For two different inertial coordinate systems moving at constant three-dimensional translational velocities relative to each other we have the inequality

$$\frac{\mathbf{B^*H^*} + \mathbf{D^*E^*}}{8\pi} \neq \frac{\mathbf{B'H'} + \mathbf{D'E'}}{8\pi}$$

where the asterisk denotes vectors that are calculated in the proper coordinate system, while the prime denotes those vectors that are determined in any arbitrary inertial coordinate system. The scalar equality

$$\frac{\mathbf{B^*H^*} + \mathbf{D^*E^*}}{8\pi} = \frac{1}{16\pi} (F_{ij} H^{ij} - 4u^{*k} u^{*s} F_k{}^j H_{sj}) \tag{2.2}$$

is, on the other hand, always valid. The four-dimensional invariant in the right-hand side of this equality provides the formula for the electromagnetic field energy in any inertial coordinate system. Components of the four-dimensional velocity of points of the proper coordinate system are denoted by $u^{*i}$ in any coordinate system in which tensor components $F_k{}^j$ and $H_{ij}$ are expressed in terms of vectors $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$.

Formula (2.2) defines the electromagnetic field energy as a four-dimensional scalar.

In the absence of a medium (i.e. in a "void") there appears at first sight an arbitrariness in the selection of the input ("proper") inertial coordinate system. Such arbitrariness is, however, only apparent, if one takes into account that the concept of tension of electric and magnetic fields is only possible when "test" bodies are brought into the field, which makes it, in turn, possible to introduce the accompanying and proper coordinate systems.

In a fixed proper inertial coordinate system formula (2.2) for the electromagnetic field energy can be applied not only at the selected point $M$ where $\mathbf{u} = \mathbf{u}^*$ but, also, at adjacent points $M'$ at which $\mathbf{u}\,(M') \neq \mathbf{u}^*\,(M')$. It is important to note that in this case $u^{*i}$ in formula (2.2) is to be understood as representing the four-dimensional velocity components of points of the proper coordinate system, which correspond to the fixed point $M$. In differentiating in formula (2.2) the energy expression with respect to the four-dimensional coordinates (particularly with respect to time) it is necessary to take into consideration equalities (1.4) which are satisfied in any coordinate system.

The equation of energy of the electromagnetic field is of the form (2.1) in any (not only in the proper) inertial coordinate system. However, in an arbitrary coordinate system the quantities $W$ and $F$ have no longer the meaning of electric field energy in the medium and of the flow of energy from the electromagnetic field to the medium, respectively. Using the Umov—Poynting equation we can represent the energy equation (2.1) in the form

$$\frac{\partial W}{\partial \tau} = \frac{1}{4\pi}\left(\mathbf{H}\frac{\partial \mathbf{B}}{\partial \tau} + \mathbf{E}\frac{\partial \mathbf{D}}{\partial \tau}\right) + \mathbf{jE} - F$$

which by simple transformations reduces to an equation of the form

$$\frac{\partial}{\partial \tau}\left(\frac{\mathbf{BH} - \mathbf{DE}}{8\pi}\right) = \frac{1}{4\pi}\left(\mathbf{H}\frac{\partial \mathbf{B}}{\partial \tau} - \mathbf{D}\frac{\partial \mathbf{E}}{\partial \tau}\right) + \mathbf{jE} - F \qquad (2.3)$$

Using the four-dimensional tensors of the electromagnetic field we introduce the four-dimensional invariants

$$\frac{\mathbf{BH} - \mathbf{DE}}{8\pi} = \frac{1}{16\pi}F_{ij}H^{ij} = L, \quad \mathbf{jE} = cI^kF_{ki}u^i \qquad (2.4)$$

$$\frac{1}{4\pi}\left(\mathbf{H}^*\frac{\partial \mathbf{B}^*}{\partial \tau} - \mathbf{D}^*\frac{\partial \mathbf{E}^*}{\partial \tau}\right) = \frac{c}{8\pi}H^{ij}u^k\nabla_k F_{ij}$$

where $\mathbf{j}^*\mathbf{E}^* = cI^kF_{ki}u^i$ is the Joule heat and $I^k$ are components of the four-dimensional electric current with components $I^\alpha = c^{-1}j^\alpha$, $I^4 = \rho_e c$.

On the basis of relations (2.4) the electromagnetic field energy equation (2.3) can be written in the invariant form

$$dL = \frac{1}{8\pi}H^{ij}\nabla_k F_{ij}dx^k + I^kF_{ki}\,dx^i - F\,d\tau \qquad (2.5)$$

where the energy flow from the medium to the electromagnetic field $F = F_4$ is determined in the proper coordinate system.

The equation of energy for a volume element of the continuous medium can be represented in the proper coordinate system in the form

$$dU = Fd\tau + dK \tag{2.6}$$

where $U$ is a four-dimensional scalar which is equal to the total energy of a volume unit taken in the proper reference system; $F$ is the energy flow from the electromagnetic field to the continuous medium, which contains Joule heat and the energy generated by the polarization and magnetization processes; $dK$ is the additional flow of external energy to a particle. We further set $dK = dQ^{(e)} - Q_i dx^i + dW_0$, where $dQ^{(e)}$ is the heat influx to the medium during the time $d\tau$, $Q_i$ is the density of the external body force, and $dW_0$ is the additional energy flow through the particle boundary and due to structural parameters. Note that when the electromagnetic field is known, equality (2.5) can be used for calculating the energy flow $Fd\tau$ from the field to the medium in terms of field characteristics. If, on the other hand, the motion of the medium is either specified or known, the energy flow $Fd\tau$ can be determined in terms of the medium motion characteristics using Eq. (2.6) and, then, substituted into (2.5).

The equation of energy for the system continuous medium-electromagnetic field in any arbitrary reference system can be written in the form

$$\begin{aligned}-d(L+U) + \frac{1}{8\pi} H^{ij}\nabla_k F_{ij}\,dx^k + I^k F_{ki}\,dx^i + \\ dQ^{(e)} + dW_0 - Q_i\,dx^i = 0\end{aligned} \tag{2.7}$$

3. **Determination of variations of tensor functions and the variational equation for the system continuous medium-electromagnetic field.** Let $\eta^A$ be the vector scalar or component of some tensor (superscript $A$ is the collective symbol of superscripts of tensors of various ranks). If two reference systems are selected, i.e. the observer's system $x^i$ and the accompanying system $\xi^k$, the tensor components $\eta^A$, which are some characteristics of the continuous medium or of the electromagnetic field, can be considered to be functions of coordinates $x^i$ or $\xi^k$.

Variations of scalar or tensor functions can be determined by various formulas. In particular, it is possible to use the variation of invariant functions as nonvariant functions that depend on the selection of the coordinate system. But it is possible to determine variations of scalars and tensors, respectively, as invariant scalar or tensor functions of coordinates with infinitely small components.

Let us consider some possible way of determining variations. Let $\eta^{\hat{}A} = \eta^{\hat{}A}(\xi^k)$ be components of some tensor (which may be a scalar) in the basis $\hat{э}_i$.

For constant $\hat{э}_i$ and $\xi^k$ we determine variations $\delta\eta^{\hat{}A}$ as infinitely small tensor components in one and the same basis $\hat{э}_i$ by formula

$$\delta\eta^{\hat{}A} = \eta^{\hat{}\prime A}(\xi^k) - \eta^{\hat{}A}(\xi^k) \tag{3.1}$$

where $\hat{\eta}^A$ and $\hat{\eta}'^A$ are components of real and varied tensors in the basis $\hat{э}_i$.

Similarly, by considering $\eta^A$ as functions of coordinates $x^i$ we determine variations $\partial\eta^A$ with constant $x^i$ and $э_i$ by the formula

$$\partial\eta^A = \eta'^A\,(x^i) - \eta^A\,(x^i) \tag{3.2}$$

where $\eta^{\wedge\prime A}$ and $\eta^A$ are components of tensors in the basis $\mathfrak{s}_i$. Note that formulas (3.1) and (3.2) determine variations $\delta\eta^{\wedge A}$ and $\partial\eta^A$ as arbitrary tensor or scalar functions of Lagrangian or Euler coordinates, respectively. This arbitrariness is related to the fact that the tensors $\eta^{\wedge\prime A}$ and $\eta'^A$ can be selected by various methods. Variations $\delta\eta^{\wedge A}$ and $\partial\eta^A$ may be considered either in bases $\mathfrak{s}^\wedge_i$ and $\mathfrak{s}_i$, respectively, or in any other basis.

Since the relation between Eulerian and Lagrangian coordinates of a point of the continuous medium represents the law of motion $x^i = x^i\,(\xi^k)$ of that medium, which is also subjected to variation, we determine the variation of the law of motion by formulas

$$\delta x^i = x'^i\,(\xi^k) - x^i\,(\xi^k)$$

The derived above variations $\delta x^i$, $\delta\eta^{\wedge A}$ and $\partial\eta^A$ can be associated at every specified point $x_0{}^i = x^i\,(\xi_0{}^k)$ by the equalities

$$\eta = \eta^{\wedge A}\mathfrak{s}^\wedge_A = \eta^A\mathfrak{s}_A$$

that follow from the definition of tensor $\eta$ and from the definition

$$\delta\eta = \partial\eta \tag{3.3}$$

where $\mathfrak{s}^\wedge_A$ and $\mathfrak{s}_A$ are polyadic products composed from vectors of bases $\mathfrak{s}^\wedge_i$ and $\mathfrak{s}_i$, respectively.

Using (3.3)and formula $\delta\mathfrak{s}^\wedge_i = \nabla^\wedge_i\delta x^s\mathfrak{s}^\wedge_s$ it is possible to write for the vector with components $\eta^{\wedge k}$ or $\eta^k$ the following series of equations:

$$\delta\eta = \delta_1\eta^{\wedge k}\mathfrak{s}^\wedge_k = (\delta_2\eta^{\wedge k} + \hat{\eta}^i\nabla^\wedge_i\delta x^{\wedge k})\,\hat{\mathfrak{s}}_k = \partial_1\eta^k\,\mathfrak{s}_k = (\partial_2\eta^k + \delta x^i\nabla_i\eta^k)\,\mathfrak{s}_k \tag{3.4}$$

where $\delta_1$, $\delta_2$, $\partial_1$ and $\partial_2$ denote various possible types of variation of components of tensor $\eta$ . From (3.4) we also obtain formulas that define the relation between these variations.

In particular, if $\mathfrak{s}^\wedge_i = \mathfrak{s}_i$, we have

$$\delta_1\eta^{\wedge k} = \partial_2\eta^k + \delta x^i\nabla_i\eta^k, \quad \delta_2\eta^{\wedge k} = \partial_1\eta^k - \eta^i\nabla_i\delta x^k \tag{3.5}$$

If $\mathfrak{s}^\wedge_i \neq \mathfrak{s}_i$, these equalities may be rewritten in the form

$$\delta_1\eta^{\wedge j}x_j{}^k = \delta_1\eta^k = \partial_2\eta^k + \delta x^i\nabla_i\eta^k \tag{3.6}$$

$$\delta_2\eta^{\wedge j}x_j{}^k = \delta_2\eta^k = \partial_1\eta^k - \eta^i\nabla_i\delta x^k$$

Similar formulas are readily formed for the covariant components $\eta_k$ with allowance for the equality $\delta\mathfrak{s}^{\wedge i} = -\nabla^\wedge_s\delta x^{\wedge i}\mathfrak{s}^{\wedge s}$. General formulas for tensors of any rank with any combination of indices are derived in the same manner.

Besides variations $\delta_i\eta^{\wedge A}$ and $\partial_i\eta^A$ $(i = 1, 2)$ which are tensor components it is possible to consider variations that are not tensor components. It is, for instance, possible to introduce variations $\delta_1{}'\eta^A$ of components of tensor $\eta^A$ in the following manner:

$$\delta_1{}'\eta^A = \partial_1\eta^A + \delta x^s\frac{\partial\eta^A}{\partial x^s} \tag{3.7}$$

where variation $\partial_1\eta^A$ has the same meaning as in formula (3.4).

Let us consider the linear scalar form

$$B_A\delta_1\eta^A + P_i\delta x^i$$

where $B_A$ are components of some tensor of the same rank as tensor $\delta\eta^A$ and $P_i$ are vector components. Depending on the investigated phenomenon it is possible to attribute to quantities $B_A$ and $P_i$ some specific physical or geometrical meaning.

Using formulas (3.4) and (3.7) it is possible to reduce the scalar form to

$$B_A\delta_1\eta^A + P_i\delta x^i = C_A\delta_1'\eta^A + Q_i\delta x^i$$

The coefficients $C_A$ and $Q_i$, unlike coefficients $B_A$ and $P_i$, are no longer components of tensors and vectors. The above considerations show that from the geometrical or physical point of view variations of the type (3.4) are preferable to those of type (3.7).

Besides the theoretically introduced variations of the investigated quantities we can consider real increments that correspond to solutions of certain problems. In the latter case variations $\delta_1$ and $\partial_1$ of parameters $\eta^A$ are replaced, in conformity with the supplementary condition, by the actual local increments of parameters $\eta^A$ using formulas

$$\delta_1\eta^A = d\eta^A = cu^i\nabla_i\eta^A d\tau = \eta^{\cdot A} d\tau, \quad \partial_2\eta^A = 0 \tag{3.8}$$

Below, we use variations $\delta_1$ and $\partial_2$ which we denote by $\delta$ and $\partial$.

The integral energy equation for a finite volume of the continuous medium cannot be derived from the energy equation considered in Sect. 2 for the system material medium-electromagnetic field, and expressed in an invariant four-dimensional form. In the special and general theory of relativity it is not generally possible to introduce the proper time and vector characteristics common to the whole body or to a finite part of it. Owing to this the equation of energy and the laws of conservation in the case of finite volumes have, generally speaking, no physical meaning. However the energy equation (2.7) provides in the case of a small volume of a continuous medium a hint of the form of terms that are to be specified for the basic variational equation.

The basic variational equation proposed by Sedov [4] is of the form

$$\delta \int_{V_4} \Lambda\, dV_4 + \delta W^* + \delta W = 0 \tag{3.9}$$

where $dV_4$ is a four-dimensional element of an arbitrary volume of the space-time $V_4$ bounded by the three-dimensional surface $\Sigma_3$, $\Lambda$ is the Lagrange function, $\delta W^*$ is the specified functional which (for continuous processes) is a volume integral taken over volume $V_4$, and $\delta W$ is a functional that represents an integral over the three-dimensional surface $\Sigma_3$. In the theory, considered here the integrand $\Lambda$ in the first term of the variational equation (3.9) is subject to variation, while the arbitrary volume $V_4$ over which integration is carried out is not varied.

The basic assumption is that the functions $L$ and $U$ for the system medium-electromagnetic field are specified as functions of the following governing parameters:

$$x_j^i,\ s,\ F_{ij},\ \nabla_k F_{ij},\ K^B \tag{3.10}$$

where $x_j^i = \partial x^i/\partial\xi^j$, $K^B$ are constant or specified functions of Lagrangian coordinates $\xi^k$ that are not varied, and $s$ is the entropy. The number $K^B$ may contain components of the metric tensor $\hat{g}_{ij}$ (when the space metric is specified) and tensor and scalar constants that define the goemetrical or physical properties of the medium.

This assumption is entirely sufficient for the construction of many important models of continuous media.

In the cases when the model construction has to define various physical effects (gyromagnetic, irreversibility of magnetization or deformation, etc.) the set of parameters of model (3.10) must be supplemented by scalar or tensor quantities $\mu^A$ (sometimes also by their derivatives) which define internal degrees of freedom of the considered physical model. The formulas and conclusions presented below are readily extended to such cases. Instead of components of tensor $F_{ij}$ we can introduce polarization and magnetization tensor components

$$P_{ij} = \frac{1}{4\pi}(H_{ij} - F_{ij})$$

which in many cases yield the same models [3-5, 7, 8].

In addition to $x_j^i$ and $F_{ij}$ we introduce in the Lagrangian $\Lambda$ components of the four-dimensional vector $A = (A^i)$ (which proves to be the field vector potential) and components of tensor $H^{ij}$, as the sought functions.

As the Lagrangian $\Lambda$ we take the sum of $-(L + U)$ which appears in the left-hand side of Eq. (2.7) as the integrand, and of the additional term $\Lambda'$. This ensures that the Maxwell equation is obtained from the variational equation, i.e. we set

$$\begin{aligned} \Lambda &= -(L + U) + \Lambda' \\ \Lambda' &= \frac{1}{8\pi} F_{ij} H^{ij} - \frac{1}{4\pi} H^{ij} \nabla_i A_j \end{aligned} \tag{3.11}$$

It is shown below that for actual processes the term $\Lambda'$ is identically zero by virtue of Euler equations.

The second law of thermodynamics for the system medium-field is of the form

$$\rho T ds = dQ^{(e)} + dQ'$$

where $s$ is the entropy per unit mass of the material medium, $dQ'$ is the uncompensated heat and $T$ is the temperature of the medium.

For simplicity we assume that the uncompensated heat is produced by two mechanisms: release of Joule heat and dissipative processes, i.e.

$$dQ' = dQ_J' + dQ_0'$$

where $dQ_J'$ is the Joule heat and $dQ_0'$ is the uncompensated heat due to irreversible processes dependent on tensor $\tau_k^{\ i}$. The irreversible effects associated with magnetization and polarization of the medium are not taken into account here.

We make a further basic assumption that for processes subject to variation the following equalities are valid (*)

$$\begin{aligned} \delta Q_I' &= I^k F_{ki} \delta x^i - I^k \partial A_k \\ \delta Q_0' &= \tau_k^{\ i} \nabla_i \delta x^k \end{aligned} \tag{3.12}$$

We use (3.12) for formulating the second law of thermodynamics for processes subjected to variation in the form

*) The presence of term $-I^k \partial A_k$ is essentially due to its use for obtaining Maxwell's equations which contain an appropriate amount of experimental data. The nonconservativeness of equations for the electromagnetic field in the presence of conduction current should be noted at this instance.

$$\rho T\,\delta s = \delta Q^{(e)} + I^k F_{ki}\delta x^i - I^k \partial A_k + \tau_k{}^i \nabla_i \delta x^k \tag{3.13}$$

The functional $\delta W^*$ represents the volume integral of the left-hand side of the energy equation (2.7) in which the term $-d(L+U)$ appearing in $\Lambda$, and the term $dW_0$, which (when $dK^B/d\tau = 0$) for processes subjected to variation becomes the functional $\delta W$, are eliminated. Variations $\delta W^*$ are obtained by substituting possible increments for actual ones, and the subsequent use of equality (3.13) in the form

$$\delta W^* = \int\limits_{V_4} \Big[\frac{1}{8\pi} H^{ij}\nabla_k F_{ij}\delta x^k + \rho T\delta s + I^k\,\partial A_k - \tau_k{}^i\nabla_i\delta x^k - Q_k\delta x^k\Big]\,dV_4 \tag{3.14}$$

4. T h e s y s t e m o f e q u a t i o n s o f m e c h a n i c s a n d **e l e c t r o d y n a m i c s.** We obtain the combined system of equations of mechanics of continuous media and electrodynamics using the variational equation (3.9), and taking the Lagrangian $\Lambda$ in the form (3.11) and the functional $\delta W^*$ in the form (3.14). We vary the first term of the variational equation (3.9) taking into account the equality

$$\delta \int\limits_{V_4} \Lambda\, dV_4 = \int\limits_{V_4} [\partial\Lambda + \delta x^k \nabla_k \Lambda)]\, dV_4$$

Assuming that variations $\partial H^{ij}$, $\partial A_k$, $\delta x^i$, $\delta s$ and $\partial F_{ij}$ are continuous and linearly independent, we obtain from the variational equation (3.9) a combined system of equations of electrodynamics and mechanics, derived by equating to zero the coefficients as independent variations in the volume integral of the variational equation (3.9). For the variations $\partial H^{ij}$ and $\partial A_k$ we obtain Maxwell's equation

$$F_{ij} = \nabla_i A_j - \nabla_j A_i \qquad \nabla_i H^{ki} = 4\pi I^k \tag{4.1}$$

For the variations $\delta x^i$ taking into account (4.3) and (4.4) we have the equations of momenta

$$\nabla_k\left(\frac{\partial (L+U)}{\partial x_j^{\,i}}\, x_j{}^k\right) - \nabla_k\Big[\frac{\partial (L+U)}{\partial \nabla_k F_{mn}}\,\nabla_i F_{mn}\Big] + \nabla_k \tau_i{}^k = Q_i \tag{4.2}$$

and for the variations $\delta s$ we have the formula for temperature $T$

$$\rho T = \frac{\partial (L+U)}{\partial s} \tag{4.3}$$

For the independent variations $\partial F_{ij}\,(i > j)$ we obtain in this case the equation of state for $H^{ij}$

$$\frac{1}{4\pi} H^{ij} = \frac{\partial (L+U)}{\partial F_{ij}} - \nabla_k \frac{\partial (L+U)}{\partial \nabla_k F_{ij}} \tag{4.4}$$

with $-F_{ij}\,(i > j)$ substituted for arguments $F_{ij}\,(i < j)$ in the function $L+U$.

If the scalar $L+U$ is independent of argument $\nabla_k F_{ij}$, the equation of state (4.4) assumes the simpler form

$$\frac{1}{4\pi} H^{ij} = \frac{\partial (L+U)}{\partial F_{ij}} \quad (i > j)$$

In addition to formulas (4.1) — (4.4) from the variational equation for the functional $\delta W$ we obtain the expression

$$\delta W = \int_{\Sigma_3} [P_i^{\ k} \delta x^i + N^{ik} \partial A_i + M^{ijk} \delta F_{ij}]\, n_k\, d\sigma_3 \tag{4.5}$$

$$P_i^{\ k} = \frac{\partial (L+U)}{\partial x_j^{\ i}} x_j^{\ k} + \tau_i^{\ k} - \frac{\partial (L+U)}{\partial \nabla_k F_{mn}} \nabla_i F_{mn} \tag{4.6}$$

$$N^{ik} = -\frac{1}{4\pi} H^{ik}, \quad M^{ijk} = \frac{\partial (L+U)}{\partial \nabla_k F_{ij}}$$

where the tensor components $P_i^k$ may be considered to be components of the tensor of energy-momentum of the system continuous medium-electromangetic field. Obviously $\nabla_k P_i^{\ k} = Q_i$.

From the Euler equations we can obtain the following relation:

$$\rho T \frac{ds}{d\tau} = \frac{dU}{d\tau} + \frac{dx^i}{d\tau} Q_i + \frac{dx^i}{d\tau} I^k F_{ki} + \frac{dx^i}{d\tau} \nabla_k S_{i\cdot}^{\cdot k} - \frac{dx^i}{d\tau} \nabla_k \tau_i^{\ k} - \tag{4.7}$$
$$\frac{\partial (L+U)}{\partial K^B} \frac{dk^B}{d\tau} - \nabla_k \left[ \frac{dx^i}{d\tau} x_j^{\ k} \frac{\partial (L+U)}{\partial x_j^{\ i}} \right]$$

if $\delta K^B = 0$, then $dK^B/d\tau = 0$; generally $\delta x^i \nabla_i K^B \neq 0$, and $S_{i\cdot}^{\cdot k}$ are components of the Minkowski tensor (see below). Equality (4.7) with allowance for the second law of thermodynamics (3.13) applied to actual processes yields, after some transformations, the equation of energy

$$\frac{d(L+U)}{d\tau} = \frac{dQ^{(e)}}{d\tau} - Q_i \frac{dx^i}{d\tau} + I^k F_{ki} \frac{dx^i}{d\tau} + \frac{1}{8\pi} [H^{jk} \nabla_i F_{jk}] \frac{dx^i}{d\tau} + \tag{4.8}$$
$$\nabla_k \left( \tau_i^{\ k} \frac{dx^i}{d\tau} \right) + \nabla_k \left[ \frac{dx^i}{d\tau} x_j^{\ k} \frac{\partial (L+U)}{\partial x_j^{\ i}} \right]$$

It will be readily seen that Eq. (4.8) is the explicit form of Eq. (2.7).

For the infinitely small transformation of coordinates $x^{i'} = x^i + \delta\eta^i$, where $\delta\eta^i$ are infinitely small functions of $x^k$, we set $\delta\eta^i = \varepsilon_j^{\ i} x^j$ in the local proper coordinate system. (The constant coefficients $\varepsilon_j^{\ i}$ that correspond to the infinitely small Lorentz transformation constitute the antisymmetric matrix $\varepsilon^{ij} = -\varepsilon^{ji}$). In this case the scalar properties of the integral

$$\int_{V_4} \Lambda\, dV_4$$

yield the relation which with the use of Euler equations can be represented in the form

$$(P^{ik} - \tau^{ik}) - (P^{ki} - \tau^{ki}) = S^{ik} - S^{ki} + 2\nabla_s \left[ \frac{\partial (L+U)}{\partial \nabla_s F_{kj}} F^i{}_j - \frac{\partial (L+U)}{\partial \nabla_s F_{ij}} F_j{}^k \right] \tag{4.9}$$

and considered as the equation of the moment of momentum for the system medium-electromagnetic field, where

$$S^{ki} = -\frac{1}{4\pi}\left(H^{ij}F_i^{\ k} - \frac{1}{4}H^{mn}F_{mn}g^{ik}\right)$$

It was assumed in all previous formulas (4.1)–(4.9) that $L$ and $U$ are given functions of arguments (3.10). For the variational equation (3.9) to convert locally into the equation of energy it is necessary for function $L$ to be defined by equality (2.4) which by virtue of (4.4) leads to the relation

$$\frac{1}{8\pi}H^{ij}F_{ij} = L = \frac{1}{2}\left[\frac{\partial(L+U)}{\partial F_{ij}}F_{ij} - \nabla_k\frac{\partial(L+U)}{\partial\nabla_k F_{ij}}F_{ij}\right](i > j) \qquad (4.10)$$

which represents a physical constraint on functions $L$ and $U$.

5. **Various models of solids, liquid, and gaseous media.** Starting with the general equations (4.1) and (4.2), and the equations of state (4.3) and (4.4) it is possible to obtain specific models of continuous media. For this it is necessary to specify functions $L$ and $U$ of the determining parameters, taking into account (4.10), and fix the physical laws that define irreversible processes for the four-dimensional current vector $I^k$ (relationships of the type of Ohm's law) and for the tensor component $\tau_i^{\ k}$ (relationships of the type of viscous stresses). In addition data on external forces $Q_k$ and on physical or geometrical parameters $K^B$ defining the macroscopic structure of the medium are, obviously, required.

The system of relations derived in this manner contains the model of a nonlinear elastic body with allowance for polarization and magnetization effects, the model of a perfect or viscous liquid or gas, as well as the model of magneto- and electro-hydrodynamics, models of ferromagnetic fluids, and many other examples of models that have been already investigated and, also, models that are still to be constructed for various classes of phenomena.

Let us consider some of the general properties of equations of state (4.4) for $H^{ij}$ and the equations of state (4.6) that reduce to the expression for tensor components of momentum energy $P_i^{\ k}$.

It is possible to assume that the function $L$ represents a quadratic form with respect to the antisymmetric tensor components $F_{ij}$:

$$L = \frac{1}{16\pi}c^{ijkl}F_{ij}F_{kl} \qquad (5.1)$$

where the tensor components $c^{ijkl}$ are some specified functions of the determining parameters $x_j^{\ i}$, entropy $s$, and also of $K^B$. Formula (5.1) implies that the tensor components $c^{ijkl}$ satisfy the equalities

$$c^{ijkl} = c^{klij} = -c^{jikl} = c^{ijlk}$$

We further assume in conformity with (4.10) that $U$ is independent of $F_{ij}$ and $\nabla_k F_{ij}$. In this case the equations of state (4.4) for the electromagnetic field assume the form

$$H^{ij} = c^{ijkl}F_{kl}, \quad c^{ijkl} = c^{ijkl}(x_q^{\ p}, s, K^B) \qquad (5.2)$$

If the tensor components $c^{ijkl}$ depend only on $g_{ij}$, $\rho$, $u^{*i}$, $s$, and possibly on some other scalar quantities, we have three-dimensional isotropy (*) and the

*) For particular forms of anisotropy other forms can be readily indicated instead of (5.3) [1].

tensor components $c^{ijkl}$ are defined by formulas

$$c^{ijkl} = \frac{1}{2}\Big[\frac{1}{\mu}(g^{ik}g^{jl} - g^{il}g^{jk}) + \left(\varepsilon - \frac{1}{\mu}\right)(g^{ik}u^{*j}u^{*l} - g^{jk}u^{*i}u^{*l} + g^{jl}u^{*i}u^{*k} - g^{il}u^{*j}u^{*k})\Big] \tag{5.3}$$

where $\varepsilon$ and $\mu$ are coefficients of permittivity and magnetic permeability which can depend on $\rho$ and $s$, or on $\rho$ and $T$, and possibly, in more general cases also on other variable or constant scalar parameters.

In the proper coordinate system in three-dimensional form formula (5.2) on the strength of (5.3) reduces to the frequently used formulas

$$\mathbf{D} = \varepsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}$$

If the material medium is neither magnetizable nor polarizable, we have $\varepsilon = 1$ and $\mu = 1$. If the medium is not magnetizable but only polarizable, $\mu = 1$ and $\varepsilon \neq 1$. Conversely, if the medium is only magnetizable but not polarizable, $\varepsilon = 1$ and $\mu \neq 1$. This simple position is inherent to the construction proposed here with the selected system (3.10) of governing parameters and supplementary assumptions.

If the scalar function $L$ and total energy $U$ are functions of the following arguments:

$$\rho, \ u^i, \ F_{ij}, \ s, \ K^B \tag{5.4}$$

then for the tensor components of the momentum energy $P_k^{\ i}$ in conformity with (4.6) we obtain

$$P_k^{\ i} = \frac{\partial(L+U)}{\partial u^j}(\delta_k^{\ j} - u_k u^j)\, u^i - \rho\frac{\partial(L+U)}{\partial\rho}(\delta_k^{\ i} - u_k u^i) + \tau_k^{\ i} \tag{5.5}$$

A specific definition of the momentum energy tensor can be obtained using supplementary assumptions of the type of (5.1) and (5.3). A further simplification of the expression for the momentum energy tensor is obtained if one assumes that

$$U = \rho c^2 + \rho U_0(\rho, \ s) \tag{5.6}$$

The continuous medium model defined by the general equations (4.1)—(4.4) with assumptions (5.3) —(5.6) is a relativistic model of a viscous isotropic compressible polarizable and magnetizable fluid after $\tau_k^{\ i}$ had been specifically defined. The momentum energy tensor which conforms to assumptions (5.3) – (5.6) is of the form

$$P_k^{\ i} = -\Big[+U + \rho^2\frac{\partial U_0}{\partial\rho}\Big](\delta^i_{\ k} - u_k u^i) + \frac{1}{8\pi}\Big[\frac{1}{2}\frac{\rho}{\mu^2}\frac{\partial\mu}{\partial\rho}F_{mn}F^{mn} + \rho\left(\frac{1}{\mu^2}\frac{\partial\mu}{\partial\rho} + \frac{\partial\varepsilon}{\partial\rho}\right)F_{mn}F_q^{\ m}u^{*n}u^{*q}\Big](\delta_k^{\ i} - u_k u^i) + \frac{1}{4\pi}\frac{\varepsilon\mu - 1}{\mu}F_{mn}F_q^{\ m}u^{*n}u^{*i}(\delta_k^{\ q} - u_k u^q) + \tau_k^{\ i}$$

The determination of components of the tensor $\tau_k^{\ i}$ related to laws of dissipation requires additional assumptions which may be different. These questions are not considered here.

The method of model construction using the basis equation (3.9) described above may, at first sight, appear fairly complicated and artificial. Its complexity is, however, associated with the essence of the matter and, generally speaking, is always present either explicitly or more often implicitly, usually without a unified and orderly system. To this it can be added that the use of variational "principles" (without $\delta W^*$ and $\delta W$) has at present become the basic and, apparently, unique source for constructing new models in the theory of relativity and other physical theories.

It should be stressed that the establishment of new physico-mechanical models is an important theoretical problem which should be investigated and resolved once for the numerous subsequent specific applications. Simplest models of perfect fluid and elastic body with their subsequent applications in hydrodynamics, aerodynamics, structural mechanics, and other fields, are established in a similar manner.

In applications and numerical solution of various specific problems it is possible to apply directly Eq. (3.9) which for the considered models contains not only the closed system of Euler equations but, also, supplementary initial and boundary conditions. Moreover, owing to the integral form of Eq. (3.9), conditions at strong discontinuities are automatically satisfied.

## REFERENCES

1. Sedov, L. I., Mechanics of a Continuous Medium, Vol. 1, 2-nd ed., Moscow, "Nauka", 1973 (see also English translation, Pergamon Press Book No. 09878, 1965).

2. Sedov, L. I., On conditions at second-order discontinuities in the theory of gravitation. PMM, Vol. 36, No. 1, 1972.

3. Sedov, L. I., On the ponderomotive forces of interaction between an electromagnetic field and an accelerating material continuum with taking into account finite deformations. PMM, Vol. 29, No. 1, 1965.

4. Sedov, L. I., Mathematical methods of constructing new models of continuous media. Uspekhi Matem. Nauk, Vol. 20, No. 5, 1965.

5. Zhelnorovich, V. A., Models of continuous media with internal electromagnetic and mechanical moments. Collection: Problem of Hydromechanics and Mechanics of Continuous Medium. Moscow, "Nauka", 1968.

6. Shtein, A. A., Models of polarizable media and the averaged relations which correspond to these in the case of high-frequency electromagnetic field. PMM, Vol. 41, No. 2, 1977.

7. Tsypkin, A. G., On a model of continuous medium with electromagnetic effects taken into account. PMM, Vol. 41, No. 1, 1977.

8. Chernyi, L. T., Construction of models of magnetoelastic continuous media with allowance for magnetic hysteresis and plastic deformations. Nauchn. Tr. Inst. Mekhaniki, MGU, No. 31, 1974.

9. Zhelnorovich, V. A. and Sedov, L. I., On the variational method of derivation of equations of state for a material medium and a gravitational field. PMM, Vol. 42, No. 5, 1978.

Translated by J. J. D.